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Numerical methods for construction reachability sets of dynamical systems

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Abstract. The research is devoted to the problem of reachability sets construction and representation in a control problem of a dynamical system. The paper discusses two numerical methods for construction reachability sets of dynamical systems. These methods differ in the way of representations of the reachability sets. The first method is oriented on solution of the control problem on the plane and connected with a representation of sets in the form of polygons. The second one is connected with a pixel representation of sets in the *m*-dimensional Euclidian space and simplicial complexes.

Keywords: Reachability sets, control problem, phase constraint, α-complex.
PACS: 07.10.12

INTRODUCTION

As a rule, the reachability sets can not be effectively described analytically. At the same time often there is a need in their calculation. Quite often this need appears in control theory, mechanics, ecology and economy fostering the development of methods and algorithms of approximate calculation of the reachability sets.

At the current moment pixel methods of reachability sets construction are widespread. In this methods space breakdown by the regular net takes place and the reachability sets are the sets of nodes of this net. This approach works fine for the low-dimensional spaces (2- or 3-dimensional). Even though it takes up to several days to compute reachability set for the quite simple examples. That is why the development of the efficient algorithms of reachability sets construction which are much faster than classic pixel methods is the essential task. This paper is about two of this efficient methods and continuous researches presented in [1, 2, 3, 4, 5, 6, 7]

PROBLEM FORMULATION

Consider the controlled system which dynamics is described by the following differential equation

$$\dot{x} = f(t,x,u),\ u \in P,\ t \in [t_0, \vartheta],\ t_0 < \vartheta < \infty. \tag{1}$$

Here, x is the m-dimensional phase vector of the system, u is the control, and P is a compact set in the Euclidian space R^r. It is assumed that standard conditions of the

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existence, uniqueness and extendability of a solution of the system (1) over time interval $[t_0, \vartheta]$ are imposed on the system:

Condition A. *The function $f(t,x,u)$ is continuous by aggregation of variables t, x, u in the domain $[t_0, \vartheta] \times R^m \times P$ and for any bounded and closed domain $D \in [t_0, \vartheta] \times R^m$ there is constant $L = L(D) \in (0, \infty)$ such, that*

$$\|f(t,x^*,u) - f(t,x_*,u)\| \le L \cdot \|x^* - x_*\|, \text{ where } (t,x_*) \in D, (t,x^*) \in D.$$

Condition B. *There exists a constant $\mu \in (0, \infty)$ such, that*

$$\|f(t,x,u)\| \le \mu \cdot (1 + \|x\|), \ (t,x,u) \in [t_0, \vartheta] \times R^m \times P.$$

Along with the system (1), compacts Φ and X_0 from $\Phi(t_0)$ are given. Here the set Φ is a phase constraint for the system (1) and it has nonempty sections $\Phi(t) = \{x \in R^m : (t,x) \in \Phi\}$, $t \in [t_0, \vartheta]$. The set X_0 plays a role of a start set. Let's consider, that sections $\Phi(t)$, $t \in [t_0, \vartheta]$, are changed continuously with a time.

Definition 1. *The reachability set $X(t^*; t_*, x_*)$, where $t_0 \le t_* < t^* < \infty$, $x_* \in \Phi(t_*)$, is a set of all $x^* \in R^m$ for which there are solutions $x[t]$ such as $x[t_*] = x_*$, $x[t^*] = x^*$, $x[t] = \Phi(t)$ where $t \in [t_*, t^*]$.*

Let's denote by $X(t^*; t_0, X_0) = \bigcup\limits_{x_0 \in X_0} X(t^*; t_0, x_0)$ a reachability set for the system (1) at time moment t^* and start set X_0.

Problem. *Necessary to construct a reachability set $X(\vartheta; t_0, X_0)$ for the system (1).*

SCHEME OF SOLUTION

Define the differential inclusion (DI) $F(t,x)$ as following

$$\dot{x} \in F(t,x), \ t \in [t_0, \vartheta], \tag{2}$$

where $F(t,x) = co\{f(t,x,u) : \ u \in P\}$.

Divide the interval $[t_0, \vartheta]$; i.e., specify the partition $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ of the interval $[t_0, \vartheta]$ such, that the diameter $\Delta = \max\{(t_{i+1} - t_i) : \ 0 \le i \le \ N-1\}$, of the partition Γ is sufficiently small.

Associate a sequence $\{\tilde{X}(t_i)\}$ of sets $\tilde{X}(t_i) \subset R^m$ with partition Γ. This sequence is defined recursively as following

$$\tilde{X}(t_0) = X_0, \ \tilde{X}(t_{i+1}) = \Phi(t_{i+1}) \cap \tilde{Z}(t_{i+1}; t_i, \tilde{X}(t_i)), \ i = 0, \ 1, \ \ldots, \ N_f - 1. \tag{3}$$

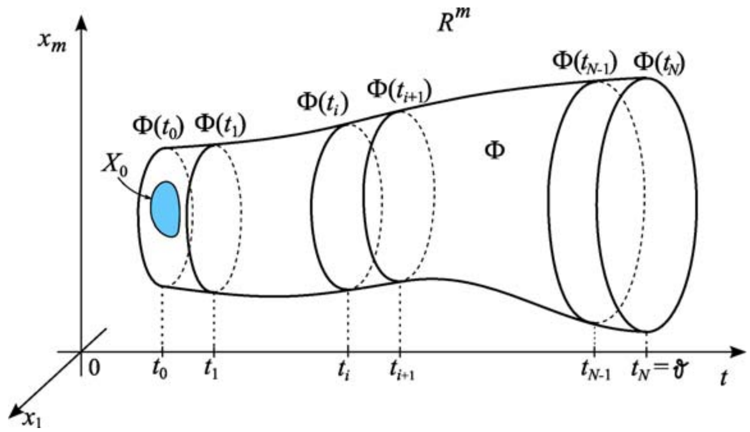

FIGURE 1. The start set X_0 and sections of the phase constraint Φ for time moments from the partition Γ.

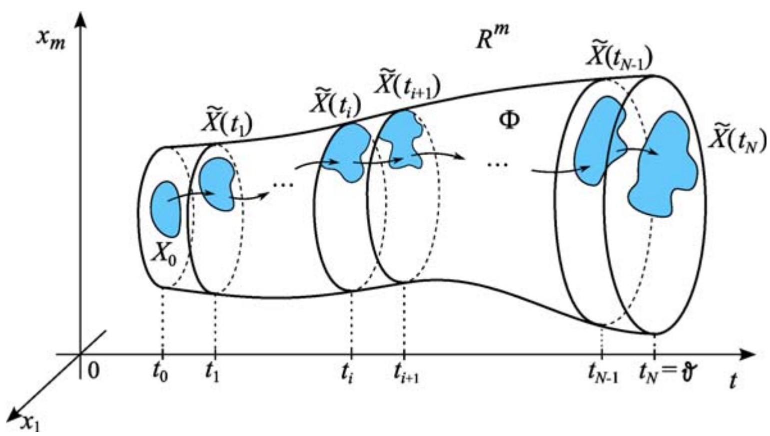

FIGURE 2. Approximations of reachability sets $\tilde{X}(t_i)$ for the time moments t_i of the partition Γ.

Here, $\tilde{Z}(t^*;t_*,x_*) = x_* + (t^* - t_*)F(t_*,x_*)$, $t_0 \le t_* < t^* \le \vartheta$, $x_* \in R^m$; $\tilde{Z}(t^*;t_*,X_*) = \bigcup\limits_{x_* \in X_*} \tilde{Z}(t^*;t_*,x_*)$.

Therefore the set $\tilde{X}(\vartheta)$ will be constructed for the time moment $t_N = \vartheta$. This set is approximation of the reachability set $X(\vartheta;t_0,X_0)$. The smaller diameter of partition Δ, the more preciously this approximation is.

NUMERICAL METHODS

Polygons method.

At this method all sets (start and final sets, reachability sets, the phase constraint) are presented as a single polygon or a union of polygons. Polygons may be non-convex. Each polygon is specified by a set of closed broken lines. One of these broken lines is an external border, others form internal border of polygon (in the common case arbitrary polygon may have number of holes). All operations of constructing reachability sets are based on operations with polygons (union, subtraction and intersection). Because of all polygons are formed by number of closed broken lines. It allows to use memory on personal computer (PC) in rational way and in many cases leads to decreased time of computations in comparison with classic grid methods. On a contrary, the polygons method has comparatively complicated logic of computations, require a very high calculation accuracy on PC and at the current realization can be applied only for the case on the plane (2-dimensional case). This method is described in details in [3].

Simplicial method.

Simplicial method is based on the pixel representation of the reachability sets. Application of this method is not related to the space discretization as in the case of the classic grid methods. Here reachability sets are presented as sets of the simplexes. Approach in which each point of reachability set is vertex of the simplex allows to exclude a lot of non-border point out of computation process. So it is important to develop algorithm of finding the border points. One of such algorithm presented in this paper is closely connected with concept of alpha-complex. This computational geometry concept was first introduced by H. Edelsbrunner in his article [8]. Alpha-complex is a subcomplex of Delaunay triangulation of a point set. Here we talk about general case of Delaunay triangulation.

Definition. *Delaunay triangulation of a point set W in m-dimensional Euclidian space is a triangulation $DT(W)$ such that no point in W is inside the circum-hypersphere of any simplex in triangulation $DT(W)$.*

Each edge or simplex of the Delaunay triangulation can be associated with characteristic radius, radius of the smallest empty ball contains this edge or simplex (empty ball is a ball which doesn't contain any point from initial set).

Definition. *For each real number α, the alpha-complex of the given point set is the simplicial complex formed by the set of edges and simplices whose radii are at most α.*

Next we'll briefly describe the algorithm of reachability sets construction based on the alpha-complexes of the point set. It consist of the next steps:

It is assumed that at the moment t_i we have set $\tilde{X}(t_i)$ which is the point set (cloud) in m-dimensional space. At the start moment t_0 we have a point cloud in m-dimensional Euclidian space which describes initial state of the system (1).

Step 1. Perform Delaunay triangulation of the point cloud $\tilde{X}(t_i)$. The result of this step is a list of edges of all simplexes of triangulation.

Step 2. Set real number $\alpha > 0$. Current algorithm implementation doesn't contain any mechanism of automatic finding of α-value. Instead of this α-value defines manually. Most of the time the α-value sets according to formula $\alpha = \Delta d$, where d is the diameter of the point cloud, suits well for real tasks. However, it is still possible to choose α-value less of greater than the value defined previously depends on a geometry of attainability sets already computed. This is some kind of a manual feedback α-value defining. Which is obviously not good, but is well enough for the first approaches. α-value in the algorithm regulates how precious our approximation is.

Step 3. Remove from the list formed on a step 1 all edges and simplexes whose circum-hypersphere radius greater then α. The result of this step is a modified list of edges.

Step 4. Remove inner edges from the triangulation (from the modified list of step 3). The edge is an inner edge if it occurs more than once in a list.

Step 5. Modified point cloud $\tilde{X}(t_i)$ formed by vertexes of the edges left after step 4 is an initial point set for the next algorithm iteration. So according to the recurrent formula (3) find the point cloud $\tilde{X}(t_{i+1})$. If $t+i \neq \vartheta$ than go step 1.

It is important to note that after performing of all steps of the algorithm it is possible to get local closed domains inside the approximated attainability set. Due to the aim of reducing calculation in some particular cases we can exclude the vertexes of this local domains from the calculation. In this case it is necessary to modify algorithm by adding the step of exclusion of a local closed domain vertexes.

In comparison with existing algorithms the algorithm based on the alpha-complexes shows easy scalability. It is already possible to use the algorithm to compute approximate attainability sets of a relatively high dimensional (up to 6) systems. Due to the exis4tence of a large number of Delaunay triangulation implementation which is core of an algorithm is possible to optimize performance significantly without any big losses in a precision. Also there are opportunities to implement the algorithm using parallel computing on the supercomputers.

It is planned to expand algorithm usage on a controlled systems with a phase constraints and on controlled systems of order up to 10.

NUMERICAL SIMULATIONS: EXAMPLES

Sample 1. The Brockett integrator.

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_1 \cdot u_2 - x_2 \cdot u_1 \end{cases}$$

where $\|u\| \leq 1$, $u \in R^2$, $x \in R^3$, $t \in [0;2]$, $\Delta = 0,1$.

Sample 2. The system of Euler equations for spinning firm body

$$\begin{cases} \dot{x}_1 = \dfrac{(j_2 - j_3)}{j_1} \cdot x_2 \cdot x_3 + u_1 \\ \dot{x}_2 = \dfrac{(j_3 - j_1)}{j_2} \cdot x_1 \cdot x_3 + u_2 \\ \dot{x}_3 = \dfrac{(j_1 - j_2)}{j_3} \cdot x_1 \cdot x_2 + u_3 \end{cases}$$

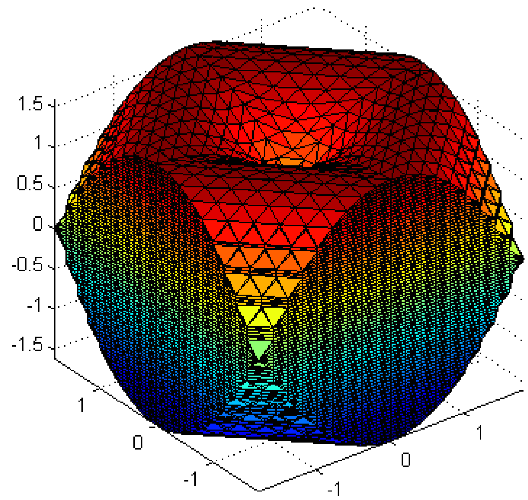

FIGURE 3. Reachability set $\tilde{X}(t)$ of the Brockett integrator at the time moment $t = 1$.

where $\|u\| \leq 1$, $u \in R^3$, $x \in R^3$, $t \in [0;1]$, $\Delta = 0,05$. Here x_1, x_2, x_3 are components of angle speed along main axis, j_1, j_2, j_3 are main insertion moments, u_1, u_2, u_3 are moments, appurtenant around main axis, they are regarded as a control

1. Start set is a point $(0,0,0)$. The reachability set $\tilde{X}(t)$ at the moment $t = 1,6$ is shown in figure 4.
2. Start set are the vertices of the cube with side length 2 and center in the origin. The reachability set $\tilde{X}(t)$ at the moment $t = 0,5$ is shown in figure 5.

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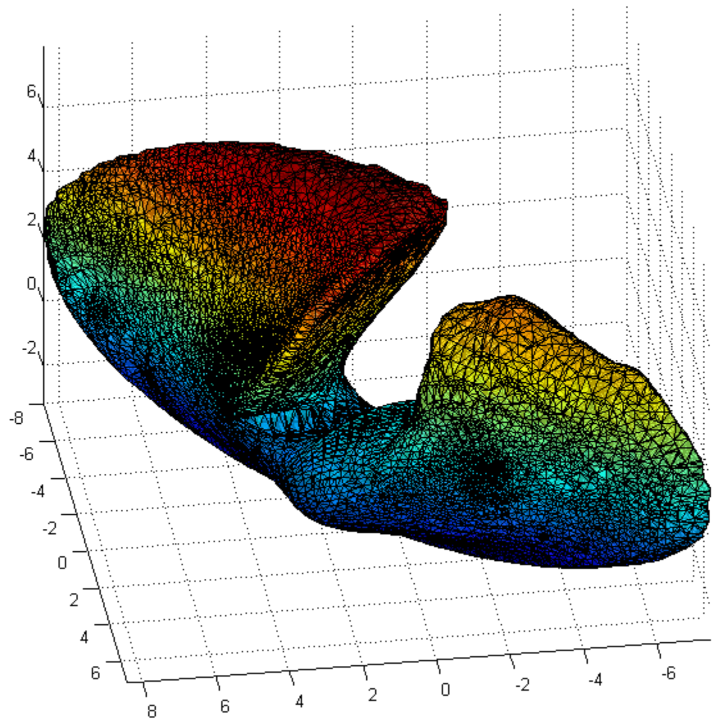

FIGURE 4. The reachability set $\tilde{X}(t)$ of Euler equations system, $X_0 = (0,0,0)$, $t = 1,6$.

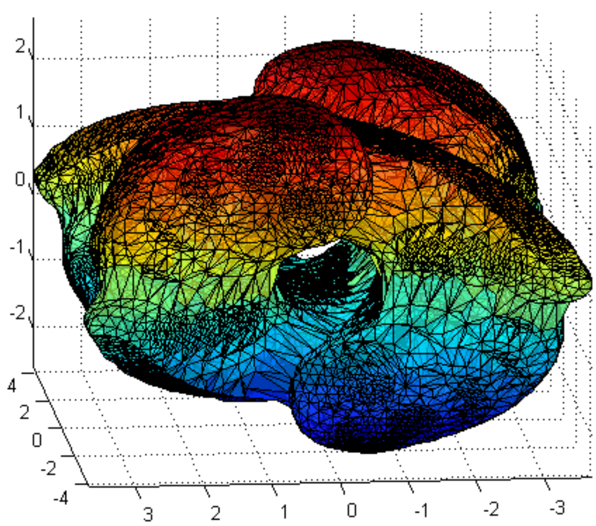

FIGURE 5. The reachability set $\tilde{X}(t)$ of the Euler equations system, X_0 — vertices of the cube, $t = 0,5$.

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